The quadruple gravitational constant, the Bronshtein cube of limits, and implications

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It is argued that the quadruple gravitational constant \(4G\) can be seen as a fundamental limit of nature. The limit holds across all gravitational systems, and distinguishes gravitationally bound from unbound systems. Describing \(4G\) as a limit allows extending the Bronshtein cube of physical theories to a cube of limits, in which each theory of modern physics, including special relativity, general relativity and relativistic quantum gravity, is described by a specific fundamental limit. The existence of these limits yields intriguing predictions for a future theory of relativistic quantum gravity.

I. INTRODUCTION: THE BRONSHTEIN CUBE

Bronshtein’s physics cube based on the speed of light \(c\), the quantum of action \(\hbar\) and the gravitational constant \(G\) has often been used to show the relation between the main physical theories [1, 2]. In the following, it is argued that the usual cube can be extended to a cube of limits, by incorporating universal and relativistic gravitation.

Special relativity and quantum theory are based on the invariant maximum speed \(c\) and the invariant minimum action \(\hbar\). In the past decades, it became clear that general relativity can be based on the invariant maximum force \(F_{\text{max}} = c^4/4G \approx 3.0 \cdot 10^{43}\) N, which is realized on gravitational horizons. It might be surprising at first that also classical gravity can be based on an invariant limit, namely the quadruple gravitational constant \(4G\), which is realized in parabolic gravitational motion. It is shown in the following that this connection allows describing all of modern physics using invariant limit quantities. In addition, it is argued that the limits strongly restrict the possible options for a future theory of relativistic quantum gravity.

II. MAXIMUM FORCE

In 1973, Elizabeth Rauscher, followed by many others, discovered that general relativity implies a maximum force \(c^4/4G\) [3–35]. In 2002, Gibbons, and Schiller in 2003, included the factor 1/4 and showed that in 3+1 dimensions the force produced by black holes is never larger than the maximum value \(c^4/4G\) [9, 10].

The maximum force value \(c^4/4G\) is due to the maximum energy per distance ratio appearing in general relativity. For example, for a Schwarzschild black hole, the ratio between the energy \(Mc^2\) and its diameter \(D = 4GM/c^2\) is given by the maximum force value, independently of the size and mass of the black hole. Also the force on a test mass that is lowered with a rope towards a gravitational horizon – whether charged, rotating or both – never exceeds the force limit, as long as the minimum size of the test mass is taken into account. All apparent counter-examples to maximum force disappear when explored in detail [23–26, 28].

In addition, maximum force \(c^4/4G\) implies Einstein’s field equations of general relativity [10, 11, 28, 29, 34]. This result, only valid in 3+1 dimensions, can be reached in two ways: it can be deduced from the curvature of space-time, and it can be deduced with the help of the first law of black hole horizons. As a result of these deductions, the maximum force limit can be seen as the defining principle of general relativity. The situation resembles special relativity, of which the maximum speed limit can be seen as the defining principle.

The maximum force principle for general relativity is not the only option. Other quantities that combine \(c\) and \(G\), such as maximum power \(c^3/4G\) [7, 14, 18, 28, 31, 32, 36–39] or maximum mass flow rate \(c^3/4G\) [27, 28], can also be taken as principles of relativistic gravitation. A similar situation arises in special relativity, where, e.g., both \(c\) and \(c^2\) can be taken as defining limit.

In short, general relativity can be deduced from the principle of maximum force \(c^4/4G\). Likewise, special relativity can be deduced from the principle of maximum speed \(c\). A question arises: can usual inverse square gravity also be deduced from a limit?

III. THE QUADRUPLE GRAVITATIONAL CONSTANT

Traditionally, the gravitational constant \(G\) is defined as the proportionality constant appearing in expressions for classical gravity, such as in the gravitational acceleration

\[
a = \frac{GM}{r^2}.
\] (1)

This expression is due to the work by Hooke, Newton, Cavendish and many others. Because the expression unified the sublunar and translunar effects, the inverse square dependence of classical gravity is often called universal gravity. Rewriting the equation using the diameter \(d = 2r\) yields

\[
a = \frac{4GM}{d^2}.
\] (2)
In the following, it will be argued that the quadruple gravitational constant $4G$ is a limit value for products of observables that yield the result $4G$ in gravitating systems.

IV. 4G AS LIMIT DISTINGUISHING UNBOUND FROM BOUND PARTICLES

A particle or test mass is unbound from a large mass $M$ if its kinetic energy is larger than the gravitational potential energy. In other terms, a particle is unbound or free if the doubled centre-to-centre distance $d = 2r$ and the speed $v$ obey

$$\frac{dv^2}{M} \geq 4G \approx 2.7 \cdot 10^{-10} \text{ m}^3/\text{kg s}^2.$$  \hspace{1cm} (3)

The constant $4G$ thus describes the difference between unbound and bound particles near a mass $M$. If a particle has a product $dv^2/M$ that is larger than $4G$, it is unbound; otherwise it is bound. For example, a rocket at double distance $d$ from the Earth's centre, flying faster than the escape velocity $v = \sqrt{4GM/d}$, is unbound. In contrast, a stone on the ground is bound to Earth.

Equivalently, a particle is unbound from a mass $M$ if

$$\frac{ad^2}{M} \geq 4G.$$  \hspace{1cm} (4)

Likewise, if a particle near a mass $M$ obeys

$$\frac{1}{\rho T^2} \geq 4G,$$  \hspace{1cm} (5)

where the ‘effective’ density $\rho = M/d^3$ and the ‘effective’ particle cycle time around the mass $T = d/v$ are implied, then the particle is unbound. In all other cases, the particle is gravitationally bound.

In short, $4G$ is the smallest possible value of any product of quantities containing a nearby mass $M$ that can arise for a gravitationally unbound particle or test mass. Crossing the limit bounds the particle or test mass to the attracting mass $M$.

Approaching the limit $4G$ from below leads to ever increasing cycle or orbital times, until parabolic motion is reached; approaching $4G$ from above leads from unbound hyperbolic to parabolic motion, when gravitational potential energy and kinetic energy are exactly equal.

V. TESTING THE LIMIT $1/4G$ IN COSMOLOGY

Using equation (5), every physical system can be described by an effective density times time squared. The equation implies that $1/4G \approx 3.7 \cdot 10^9 \text{ kg s}^2/\text{m}^3$ is the largest possible value for effective density times time squared that can arise in an unbound system.

This also applies to the universe as a whole. The quantity $1/4G$ limits the product $\rho T_H^2$ of matter density and (Hubble) time squared. Present data [40] shows that the limit is not exceeded. One thus can state that the matter in the universe, seen at a large scale, is generally unbound or at the limit between bound and unbound state.

In short, the limit $4G$ holds in cosmology.

VI. WHY IS THE LIMIT $4G$ AND NOT SIMPLY $G$?

The factor 4 arises because of the historical preference of using the radius over the diameter in the definition of gravitationally bound systems. The conventional use of the radius has a disadvantage: it leads to different multiples of $G$ in equations (2) to (5). This does not happen if the diameter is used instead.

Because of the historical preference for radius over diameter, the factor 4 also appears in the maximum force $c^4/4G$. Furthermore, as shown in reference [28], the factor 4 in maximum force is also the origin of the factor 4 in the expression $S/k = Ac^4/4G\hbar$ for black hole entropy.

VII. TESTING THE LIMIT $4G$ IN ROTATING GALAXIES

There is one group of situations where the validity of the limit $4G$ is uncertain. The rotation of stars orbiting galaxy centers is an intense topic of research. In almost all galaxies the most distant stars are measured to rotate faster than predicted from inverse square gravity with the estimated central mass values. Different explanations have been proposed.

In the most common explanation, the deviation is explained with yet unobserved cold dark matter [40]. A minority of researchers explores deviations from the inverse square dependence of gravitation [41]. They postulate that at distances that would lead to accelerations smaller than a universal constant $a_0 \approx 1.2 \cdot 10^{-10} \text{ m/s}^2$, the actual acceleration due to gravitation is larger than the one predicted by inverse square gravity.

Both explanations can explain the observed rotation curves. The explanation with dark matter claims that stars at the outer edge of galaxies are bound because the actual mass is larger than the luminous mass. Theories with deviations from inverse square gravity claim that at large distances, masses remain bound even if speeds are larger than the conventional escape velocity.

The discussion between the two explanations – and a number of additional ones – is not settled yet. It might even be, for example, that the constant $a_0$, if it exists, is due to some quantum effect on the cosmological scale, so that the limit $4G$ remains valid. Future research will show which explanation is correct.

In short, so far, there is no definite observation contradicting the limit $4G$. 
FIG. 1: The Bronshtein limit cube of physical theories is illustrated, in which every physical theory is described by a limit. Following the lines towards the top leads to an increase of the precision of the description of nature. As explained in the text, at each corner, other choices of limits – upper or lower – are also possible.

VIII. PROPERTIES AND LIMITATIONS OF THE LIMIT $4G$

The limit $4G$ is *invariant*. Boosts do not change the limit value. The same applies to $\hbar$, $c$ and $c^4/4G$.

Like $c$, $\hbar$ and $c^4/4G$, also the limit $4G$ applies only to *real and free* particles. Bound or virtual particles do not comply with the limits.

Like $c$ and $c^4/4G$, also the limit $4G$ applies only to observables measured at a single location in space-time. It does not apply to any sum of observables at different positions, such as any sum of observables for different particles in a gas, or sums of observables at different times, or both. All these limits are *local*.

IX. INVERSE SQUARE GRAVITY

To complete the parallel to special and general relativity, it makes sense to recall that a lower limit $4G$ implies universal gravity. Indeed, inverse square gravity is implicit in the definition of $4G$.

Fans of pointed formulations could take this analogy to the extreme and call the limit value $4G$ a *principle* of classical gravitation. However, the analogy to the other theories of physics is best drawn with the help of *limits*, as argued in the following.
X. THE BRONSHTEIN LIMIT CUBE OF PHYSICAL THEORIES

Since almost a hundred years, the Bronshtein cube, illustrated in Figure 1, is used to structure the different parts of physics [1, 2]. Starting at the lowest point, Galilean physics, the descriptions of nature get more and more precise towards the top of the cube.

The results for universal gravity and for general relativity explored above now allow defining a limit at every corner of the cube – except for Galilean physics. The origin of this possibility is the following summary of modern physics:

- Universal gravity is equivalent to \( dv^2/M \geq 4G \).
- Special relativity is equivalent to \( v \leq c \).
- General relativity is equivalent to \( F \leq c^4/4G \).
- Quantum theory is equivalent to \( W \geq \hbar \).
- Quantum field theory is equivalent to \( m_\ell \geq \hbar/c \).
- Non-relativistic quantum gravity is equivalent to free particles with \( A\ell^3 \geq 4G\hbar \).
- Relativistic quantum gravity is equivalent to \( l^2 \geq 4G\hbar/c^3 \).

Here, \( l \) is length, \( W \) is action, and \( A \) is area. Each of the seven limit expressions is associated to one of the upper corners of Bronshtein’s physics cube. The result is illustrated in Figure 1. As mentioned, the choice of the limit for every field of physics is not unique; at each corner of the cube, each nonzero exponent of \( 4G, c \) and \( \hbar \) can be changed at leisure. This allows assigning various equivalent upper or lower limits at each corner.

A number of properties of this Bronshtein limit cube are worth pointing out. First, whenever a limit \( c, 4G \) or \( \hbar \) is added to a given description of motion, a more precise description is obtained. It is thus expected that relativistic quantum gravity, even though not known in all details, will be the most precise description possible – because it takes into account all limits. Among others, it limits length by \( l \geq \sqrt{4G\hbar/c^3} \), or twice the Planck length.

Secondly, apart from the correcting factor 4, all limits are Planck limits. In other terms, one can say that the (corrected) Planck limits define modern physics.

Thirdly, if desired, the lower limit for entropy, defined by the Boltzmann constant \( k \) – or another thermodynamic variable – can be added to the discussion [1, 42].

In short, all fields of physics can be defined with limits. This characterization is also useful for the teaching of physics. Learning physics starts at the bottom of the Bronshtein cube, where no limits are assumed, and proceeds towards the top, where all physical observables are limited by the corrected Planck values.

In addition, the limits of nature imply several consequences for the complete description of nature.

XI. PREDICTIONS ABOUT RELATIVISTIC QUANTUM GRAVITY

As long as rotating galaxy curves do not disprove inverse square gravity, the lower limit property of the quadruple gravitational constant \( 4G \) allows defining every theory of modern physics using a limit on the motion of unbound particles. This includes the still elusive theory of relativistic quantum gravity.

As long as rotating galaxy curves do not disprove inverse square gravity, each theory at each corner of the Bronshtein limit cube agrees with all observations.

The simplicity, the agreement with experiment and the explanatory power of the fundamental limits at each corner of the Bronshtein cube are fascinating. In addition, the limit cube implies several testable predictions.

First, the Bronshtein cube suggests that there is no physics beyond general relativity, even for the strongest gravitational fields. E.g., this implies the lack of terms that modify the Hilbert action. The cube also suggests that there is no physics beyond special relativity, even for the largest speeds or for the fastest processes. This implies the lack of doubly special relativity. Furthermore, the cube suggests that there is no physics beyond quantum theory, even for the smallest processes. This implies the lack of additional dimensions, of discrete space-time, and of non-commutative space-time.

These predictions agree with all observations so far. In fact, these predictions eliminate various candidate theories proposed in the research literature.

Describing nature with limits implies that there are no trans-Planckian effects in nature. In particular, the limits imply:

- There are no points in space or instants in time.
- Space-time is effectively continuous, despite the existence of a smallest length \( \sqrt{4G\hbar/c^3} \).
- There are no higher dimensions.
- There are no lower dimensions.
- There is no discrete space-time.
- There is no space-time foam.
- There is no space-time lattice.
- There are no singularities.

The Bronshtein cube with limits also shows that physics laws are simple. Every theory in physics can be defined with a limit only. Every theory of physics is based on a mathematically simple limit. This suggests that also the theory of relativistic quantum gravity is based on a mathematically simple limit, such as the smallest length \( \sqrt{4G\hbar/c^3} \). Like for the other theories, there is no unique defining limit; any corrected Planck value that contains \( 4G, \hbar \) and \( c \) can be used. This includes a minimum time, a minimum area, a minimum volume, a maximum acceleration, a maximum mass density, etc.

In the Bronshtein cube of limits, all lines converge to the relativistic quantum gravity. The convergence implies that the length limit \( \sqrt{4G\hbar/c^3} \) (or any equivalent limit), alone by itself, describes and implies relativistic
quantum gravity. The Bronshtein cube thus suggests that the complete description of motion is essentially known already. Whatever the microscopic degrees of freedom of space, matter and radiation may be, they follow the corrected Planck limits.

On the other hand, several unsolved issues do not appear in the cube: the origin of particle masses, gauge coupling constants and mixing angles do not arise at all. The Bronshtein cube thus on one hand provides no clue at all for understanding the fundamental constants, but on the other hand provides hope that relativistic quantum gravity, the complete description of motion, is near. Like for all predictions in physics, the future will tell.