From maximum force via the hoop conjecture to inverse square gravity

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The equivalence of maximum force $c^4/4G$ and the field equations of general relativity provides a simple derivation of inverse square gravity. The derivation confirms the hoop conjecture and suggests a lack of gravitational physics beyond general relativity. Possible loopholes are pointed out.

Introduction: maximum force

The discovery of the inverse square dependence of gravity by Hooke and Newton unified the description of motion in the sublunar and the translunar realms. Later, Einstein's special and general theories of relativity unified the description of motion for large speeds, for highly concentrated masses, and for the universe at large. Special relativity is based on the invariant maximum speed c that is realized by massless radiation. Similarly, general relativity can be based on the invariant maximum force

$$F_{\max} = \frac{c^4}{4G} \approx 3.0 \cdot 10^{43}\,\mathrm{N} \quad , \tag{1}$$

that is realized by gravitational horizons. Here, G is the gravitational constant. Interestingly, maximum force allows deducing inverse square gravity in a quick way.

It is known since 1973 that *general relativity contains and implies a maximum force* [1–21]. As Gibbons showed, the force between two black holes is never larger than the maximum value, including the factor $1/4$ [7]. The maximum force value $c^4/4G$ arises because there is a maximum energy per distance in general relativity: the ratio between the energy Mc^2 of a Schwarzschild black hole and its diameter $D = 4GM/c^2$ is given by the maximum force value, independently of the size and mass of the black hole. Other types of black holes – whether charged, rotating or both – do not allow producing larger ratios. Also the force on a test mass that is lowered towards a gravitational horizon with a string never exceeds the force limit, when the minimum size of the test mass is taken into account. No physical system allows exceeding the force limit. Maximum force passes all known experimental and theoretical tests. All apparent counter-examples to maximum force disappear when explored in detail [22–26].

Maximum force $c^4/4G$ *implies and contains* Einstein's field equations of general relativity. There are at least two ways to show this [8, 9, 26, 27]. One way starts by showing that maximum force implies a limit on the elastic deformation of space. This limit implies a relation between energy and curvature, which then implies the field equations. The other way uses maximum force to deduce the first law of horizon mechanics, which in turn implies the field equations.

Given that the field equations follow from maximum force, one can see maximum force $c^4/4G$ as a *principle* of nature. Maximum force in general relativity can be compared to maximum speed in special relativity.

Given that maximum force implies Einstein's field equations, it must imply inverse square gravity. Interestingly, the derivation is rather simple.

A new derivation of inverse square gravity

The definition of maximum force $c^4/4G$ as energy per length can be rephrased by stating that there is a maximum energy in any enclosed spherical area A with circumference πD. This is the hoop conjecture [28, 29]:

$$F_{\max} = \frac{c^4}{4G} \geq \frac{E}{A}\,\pi D \quad . \tag{2}$$

In special relativity, the acceleration a of a test mass around a sphere of diameter D is limited by

$$a \leq \frac{c^2}{D} \quad . \tag{3}$$

The two limits describe the same situation. Setting them equal to eliminate D yields

$$E = \frac{c^2}{4\pi G}\, a\, A \quad . \tag{4}$$

This consequence of maximum force is, at the same time, a version of the first law of horizon mechanics [30, 31]. Inserting the relations $E = Mc^2$ and $A = 4\pi r^2$ – valid for flat space and thus away from any horizon – yields

$$a = \frac{MG}{r^2} \quad . \tag{5}$$

Thus inverse square gravity is, in flat space, a direct consequence of maximum force and maximum speed.

This derivation of inverse square gravity from general relativity is simpler than the derivation usually found in textbooks. The derivation completely avoids the use of tensors. Therefore, this derivation can be useful in teaching.

Limitations of the derivation in alternative theories of gravity

In alternative theories of gravity, maximum force usually does not hold, or is not an invariant independent of mass, or has

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a different value. Moffat's theory is an example where the maximum force depends on an additional parameter [19]. In fact, most alternative theories of gravity that contradict experiments also appear to contradict the force limit $c^4/4G$, and vice versa [12, 19, 20, 32–35].

However, there are at least two non-trivial cases. The first is Brans-Dicke theory. As discussed by Barrow and Dadhich [19], in Brans-Dicke theory, Schwarzschild black holes and vacuum solutions are indistinguishable from those in general relativity. If the newly introduced scalar field is *weak*, it is impossible to distinguish Brans-Dicke theory from general relativity – using maximum force only – also for non-vacuum solutions. In the above derivation of inverse square gravity, a deviation for weak fields amounts to having an effective deviation from the expression $A = 4\pi r^2$ for large values of r.

Secondly, the impossibility to distinguish a theory from general relativity – using maximum force only – also seems to apply to Milgrom's modified Newtonian gravity (MOND) [36]. Also this alternative to general relativity only changes the extremely weak field case, when distances are of the order of a galactic radius. Again, for such large distances, the expression $A = 4\pi r^2$ might not be applicable in the above derivation. Because MOND provides an alternative to dark matter, it is of special interest; its relation to maximum force will be explored in detail in a future paper.

In short, the above derivation of inverse square gravity from maximum force leaves a door that is slightly open for deviations from general relativity, but only at extremely weak gravitational fields.

Testable predictions

Several testable predictions for research on gravitation follow from maximum force. First, maximum force implies that the hoop conjecture is valid; both concepts are closely tied to horizons.

Secondly, the simplicity of the principle of maximum force suggests that *no deviations* from general relativity will ever be found, in particular for strong fields. Indeed, the most recent observations about black hole mergers [37] and about the double radio pulsar PSR J0737–3039A/B [38] failed to find any deviation.

Thirdly, defining gravity and general relativity using the force limit $F_{\max} = c^4/4G$ forms a *limit triplet* together with the definition of special relativity using the speed limit $v_{\max} = c$ and that of quantum theory using the action limit $W_{\min} = \hbar$. The limit triplet predicts the lack of any trans-Planckian effect in nature, both macroscopic and microscopic. So far, no such effect was observed, despite intense searches in cosmology, black hole physics and particle physics. Among many other limits, maximum force implies the existence of a smallest length.

Finally, the explanatory power and the simplicity of the limit triplet suggest that it should also hold in any unified theory. This prediction can be tested in the future.

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